

Dualities in integrable systems: geometrical aspects

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Abstract

We discuss geometrical aspects of different dualities in the integrable systems of the Hitchin type and its various generalizations. It is shown that T duality known in the string theory context is related to the separation of variables procedure in dynamical system. We argue that there are analogues of S duality as well as mirror symmetry in the many-body systems of Hitchin type. The different approaches to the double elliptic systems are unified using the geometry behind the Mukai-Odesskii algebra.

1 Introduction

During the last years duality becomes a very fashionable term denoting a lot of different phenomena mainly due to the dramatic development in the string theory. Hence it is necessary to be very precise when speaking on the related issues. That is why we formulate the subject of the paper from the very beginning - the different dualities in the Hitchin type many-body systems as well as their generalizations shall be discussed. These include Gaudin system, Calogero and Toda type systems, their relativizations due to Ruijsenaars and more general systems involving the elliptic dependence on the momentum variables including the double elliptic model.

The idea of some hidden duality property which could help to solve these systems was pioneered by Ruijsenaars in [1]. It has been actually forgotten for a decade and was revived recently after the breakthrough in the string theory. In the recent works [2, 3] it was reformulated in more geometric terms utilizing the fact that the phase spaces of

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the corresponding systems are the manifolds coinciding with some moduli spaces. These moduli spaces or their close relatives enjoy a large symmetry group due to their origin and we shall demonstrate that part of the symmetries can be precisely formulated in terms of duality.

Seemingly different problem concerns the separation of variables procedure which means that the complicated interacting many-body problems can be reduced to the collection of the identical systems with one degree of freedom. The technical recipe for the derivation of the separated variables in a wide range of the dynamical systems was formulated by Sklyanin [4]. Recently it was shown [3] that the separation of variables for the Hitchin type many body systems can be recognized within the duality group. It appeared that this procedure has the T duality transformation as the string theory counterpart.

All types of dualities can be thought of as a kind of canonical transformation therefore the detailed structure of the phase spaces has to be investigated to get the complete set of the independent symmetries. In the first type of duality the initial action-angle variables in the initial system become p, q variables in the dual one. In the separation of variables procedure one maps the initial phase space into the symmetric power of the symplectic manifold which serves as the phase space for the one-body problem. Finally one more duality which is analogue of the S duality in the string theory can be formulated [2] as the change of the basis in the base of the Liouville tori fibration over the space of the integrals of motion.

The phase spaces of the integrable systems we are interested in admit several descriptions. The first one involves the Hamiltonian reduction procedure for the group like symplectic manifolds. The group approach to the integrable many body system is known for a while [5] and the idea to use the Hamiltonian reduction procedure for the finite dimensional groups to derive the Calogero type systems was invented in [6]. The generalization along this line involves the affine groups and algebras [7, 8] which amounts to the more general Calogero and Ruijsenaars systems while the double affine structures amount to the elliptic Calogero and Ruijsenaars models [9, 10]. The part of such systems can be considered as examples of the Hitchin systems [11].

Being reformulated in terms of the gauge theories this approach yields the effective description of the dynamics on the different moduli spaces. Moduli space involved can be identified as the moduli space of the flat G connections on the torus with marked points, moduli space of the holomorphic vector bundles on the Riemann surfaces and moduli spaces of instantons and monopoles. It appears that essentially all relevant moduli spaces can be produced from the instanton moduli spaces moreover generically instantons have to be considered on the noncommutative manifolds.

Another line of reasoning involves the idea of the separation of the variables. Within this approach the integrable systems differ by the four dimensional symplectic manifolds where the separated variables live on. Then the dynamics is reformulated in terms of the ADHM like description of the Hilbert scheme of points on the surface. The simplest models like rational and trigonometric Calogero systems can be effectively considered along this approach however for the more complicated systems there is the obstacle due to the lack of the ADHM like description of points on the generic four dimensional manifolds.

Nevertheless we shall formulate very explicitly the corresponding four dimensional manifolds for all models including double elliptic models. We shall unify two seemingly different approaches to double elliptic two-body system [2, 12] utilizing the Mukai-Odesskii algebra associated with this manifold.

This paper is an extended version of the V.R. talk given during the Kiev NATO Advanced Research Workshop "Dynamical Symmetries of Integrable Quantum Field Theories and Lattice models" in September 2000. We are based heavily on our papers [2, 3] but we also present some new results which are part of our joint works with H. Braden and A. Odesskii in progress.

The work is organized as follows. In section 2 we describe the algebraic approach based on the Hamiltonian reduction procedure and formulate the analogue of the mirror symmetry. Some comments concerning S duality transformation and the quantum case are presented. In section 3 we explain the geometry behind the separation of the variables procedure and clarify the corresponding four dimensional manifolds for all systems under consideration. Some new results concerning the double elliptic system and its possible generalizations are presented. In Section 4 we conclude with some remarks concerning the application of these dualities to the physical models.

2 "Mirror" symmetry in dynamical systems

The first type of duality we would like to discuss concerns the dualities between the pair of dynamical systems [1, 2]. It has been formulated by Ruijsenaars [1] in the context of the transition to the action-angle variables. In [2] the general procedure for the analogous symmetry within integrable system in terms of Hamiltonian and Poissonian reductions was formulated. Symmetry maps one dynamical system with coordinates x_i to another one whose coordinates coincide with the action variables of the initial system and vice versa. It is essential that transition from the Hamiltonians in the initial dynamical system to the ones in the dual system can be also formulated in geometrical terms.

Qualitatively this symmetry is even more transparent in terms of separated variables. The proper object to discuss in the separated variables is the hyperkahler four dimensional manifold which provides the phase space for one degree of freedom. In the most general situation under consideration the manifold involves two tori or elliptically fibered K3 manifold. One torus provides momenta while the second yields the coordinates. The duality at hands actually interchanges momentum and coordinate tori and in some case self-duality is expected. Since the manifold typically has the structure of the bundle the interchange of the base and the fiber is highly nontrivial operation. All other cases correspond to some degeneration.

We will discuss mainly classical case with only few comments on the quantum picture. Since the wave functions in the Hitchin like systems can be identified with some solutions to the KZ or qKZ equations the quantum duality would mean some relation between solutions to the rational, trigonometric or elliptic qKZ equations. Recently the proper symmetries for KZ and qKZ equations were found in [14]. To find the proper symmetry

2.1 Two-body system ($SU(2)$)

Let us discuss first two-body system corresponding to $SU(2)$ case. Two-particle systems which we are going to consider reduce to a one-dimensional problem. The action-angle variables can be written explicitly and the dual system emerges immediately once the natural Hamiltonians are chosen. The problem is the following. Suppose the phase space is coordinatized by (p, q) . The dual Hamiltonian is a function of q expressed in terms of I, φ , where I, φ are the action-angle variables of the original system : $H_D(I, \varphi) = H_D(q)$.

Consider as example elliptic Calogero model which is rather illustrative since already in this case the duality provides an important tool to generate new integrable systems with the elliptic dependence on momenta. The Hamiltonian reads:

$$H(p, q) = \frac{p^2}{2} + \nu^2 \wp_\tau(q). \quad (1)$$

Here p, q are complex, $\wp_\tau(q)$ is the Weierstrass function on the elliptic curve E_τ :

$$\wp_\tau(q) = \frac{1}{q^2} + \sum_{\substack{(m, n) \in Z^2 \\ (m, n) \neq (0, 0)}} \frac{1}{(q + m\pi + n\tau\pi)^2} - \frac{1}{(m\pi + n\tau\pi)^2} \quad (2)$$

Let us introduce the Weierstrass notations: $x = \wp_\tau(q)$, $y = \wp_\tau(q)'$. We have an equation defining the curve E_τ :

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) = 4 \prod_{i=1}^3 (x - e_i), \quad \sum_{i=1}^3 e_i = 0 \quad (3)$$

The holomorphic differential dq on E_τ equals $dq = dx/y$. Introduce the variable $e_0 = 2E/\nu^2$. The action variable is one of the periods of the differential $\frac{p dq}{2\pi}$ on the curve $E = H(p, q)$:

$$I = \frac{1}{2\pi} \oint_A \sqrt{2(E - \nu^2 \wp_\tau(q))} = \frac{1}{4\pi i} \oint_A \frac{dx \sqrt{x - e_0}}{\sqrt{(x - e_1)(x - e_2)(x - e_3)}} \quad (4)$$

The angle variable can be determined from the condition $dp \wedge dq = dI \wedge d\varphi$:

$$d\varphi = \frac{1}{2iT(E)} \frac{dx}{\sqrt{\prod_{i=0}^3 (x - e_i)}} \quad (5)$$

where $T(E)$ normalizes $d\varphi$ in such a way that the A period of $d\varphi$ is equal to 2π :

$$T(E) = \frac{1}{4\pi i} \oint_A \frac{dx}{\sqrt{\prod_{i=0}^3 (x - e_i)}} \quad (6)$$

Thus,

$$2iT(E)d\varphi = \frac{dx}{\sqrt{4\prod_{i=0}^3(x-e_i)}} \quad (7)$$

$$\omega d\varphi = \frac{dt}{\sqrt{4\prod_{i=1}^3(t-t_i)}} \quad (8)$$

where

$$\omega = -2iT(E)\sqrt{e_{01}e_{02}e_{03}} = \frac{1}{2\pi} \oint_A \frac{dt}{\sqrt{4\prod_{i=1}^3(t-t_i)}} \quad (9)$$

$$t = \frac{1}{x-e_0} + \frac{1}{3} \sum_{i=1}^3 \frac{1}{e_{0i}}; t_i = \frac{1}{3} \sum_{j=1}^3 \frac{e_{ji}}{e_{0i}e_{0j}} \quad (10)$$

where $e_{ij} = e_i - e_j$

Introduce a meromorphic function on E_τ :

$$\widehat{cn}_\tau(z) = \sqrt{\frac{x-e_1}{x-e_3}} \quad (11)$$

where z has periods 2π and $2\pi\tau$.

Then we have:

$$H_D(I, \varphi) = \widehat{cn}_\tau(z) = \widehat{cn}_{\tau_E}(\varphi) \sqrt{1 - \frac{\nu^2 e_{13}}{2E - \nu^2 e_3}} \quad (12)$$

where τ_E is the modular parameter of the relevant spectral curve $v^2 = 4\prod_{i=1}^3(t-t_i)$:

$$\tau_E = \left(\oint_B \frac{dt}{\sqrt{4\prod_{i=1}^3(t-t_i)}} \right) / \left(\oint_A \frac{dt}{\sqrt{4\prod_{i=1}^3(t-t_i)}} \right). \quad (13)$$

For large I , $2E(I) \sim I^2$.

Therefore the elliptic Calogero model with rational dependence on momentum and elliptic on coordinate maps into the "mirror" dual system with elliptic dependence on momentum and rational on coordinate. The generalization to the elliptic Ruijsenaars model is quite straightforward [2, 12] and the system dual to it manifests the elliptic dependence on the momentum and trigonometric on coordinate.

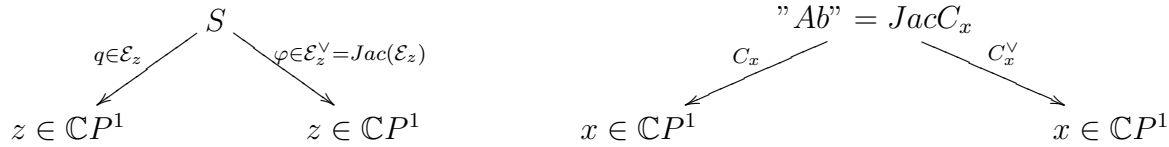
The most general system involves elliptic dependence both on coordinate and momenta. The Hamiltonian of the double elliptic system in the form

$$H(p, q) = \alpha(q|k) cn(p\beta(q|k, \tilde{k}) | \frac{\tilde{k}\alpha(q|k)}{\beta(q|k)}) \quad (14)$$

where $\alpha(q|k) = \sqrt{1 - \frac{g^2}{sn^2(q|k)}}$ and $\beta(q|k, \tilde{k}) = \sqrt{1 - \frac{g^2 \tilde{k}}{sn^2(q|k)}}$, has been found using the self-duality argument in [12].

It is instructive to compare the Hamiltonian above with the double elliptic system suggested in [2]. The four dimensional manifold which provides the phase space in that paper was identified with the elliptically fibered K3 manifold. The suggested Hamiltonian was linear in the coordinate on the base of the fibration. The Hamiltonian of the dual system was linear in the coordinate on the fiber. We shall see later that being considered in the separated variables double elliptic system involves the Mukai-Odesskii algebra on the intersection of quadrics. With the natural Poisson bracket algebra on this manifold the Hamiltonian written in the Darboux variables above exactly coincides with the coordinate on the base of the elliptic fibration while the dual Hamiltonian is linear in the coordinate of the fiber. Therefore two forms of the double elliptic system are essentially equivalent.

This approach dictates that the manifold in [12] is the double covering of the elliptically fibered rational surface. Moreover in these terms the Hamiltonian of the direct system is the coordinate on the fiber while the one for the dual system with the coordinate on the base. Therefore we could consider the same arguments for the double elliptic system from [2] in the case of the rational elliptic fibrations, (incomplete) hyperelliptic curves of genus 2 (a typical fiber of the dual system) and the double cover looks like a hyperelliptic Jacobian of the genus 2 curve (this gives us a direct imbedding of our picture in the Beauville - Mukai systems a la K. Takasaki [13]). We can illustrate the situation with the following diagramme of double elliptic system with the interchange of the direct and dual systems.



Moreover , now we are able to show the coincidence of the "naive" definition of the duality originated in the "duality relation" of [12] arising from the anticanonicity condition:

$$dP \wedge dQ = -dp \wedge dq.$$

We consider two "dual" elliptic fibrations associated with the double elliptic system of [12] system and with its dual as projective Weierstrass cubic families in \mathbb{CP}^2 :

$$S \subset \mathbb{CP}^2 : y^2 = z^3 + f(x)z + g(x), x \in \mathbb{CP}^1,$$

and z, x are nonhomogeneous coordinates in \mathbb{CP}^2 , with the canonical symplectic form $\Omega = \frac{dz \wedge dx}{y}$ and

$$\tilde{S} \subset \mathbb{CP}^2 : y^2 = x^3 + \tilde{f}(z)x + \tilde{g}(z), z \in \mathbb{CP}^1,$$

and x, z are nonhomogeneous coordinates in \mathbb{CP}^2 , with the canonical symplectic form $\Omega = \frac{dx \wedge dz}{y}$.

Now the local action coordinate $I(x)$ is computed (as we had argued in [2]) as

$$dI(x) = T(x)dx = \left(1/2\pi \oint_{A_x} dz/y \right) dx,$$

where $[A_x] \in H_1(\mathcal{C}_x, \mathbb{Z})$ is a chosen A -cycle. Analogously, the dual action coordinate $I^D(z)$ is satisfied to

$$dI^D(z) = T^D(z)dz = \left(1/2\pi \oint_{L_z} dx/y\right) dz,$$

where $[L_z] \in H_1(\mathcal{C}_z, \mathbb{Z})$.

Now we have the following chain of transformations:

$$\Omega = \frac{dz \wedge dx}{y} = -\frac{dx \wedge dz}{y},$$

$$\frac{dz}{y} \wedge \frac{dI}{T(x)} = -\frac{dx}{y} \wedge \frac{dI^D}{T^D(z)},$$

or

$$\frac{dz}{yT(x)} \wedge dI = -\frac{dx}{yT^D(z)} \wedge dI^D.$$

Now (using the expressions for the angle variables $\varphi = \frac{dz}{yT(x)}$ and $\varphi^D = \frac{dx}{yT^D(z)}$) we obtain

$$d\varphi \wedge dI = -d\varphi^D \wedge dI^D.$$

Keeping in mind the interpretation of [12], we have from the equality above

$$K(k)dp^{Jac} \wedge dI = -K(\tilde{k})dp^{Jac^\vee} \wedge dI^D,$$

or

$$dP \wedge dQ = -dp \wedge dq$$

as it was supposed in [12]!

2.2 Many-body systems

Now we would like to demonstrate how the "mirror" transform can be formulated in terms of Hamiltonian or Poissonian reduction procedure. It appears that it corresponds in some sense to the simultaneous change of the gauge fixing and Hamiltonians. More clear meaning of these words will be clear from the examples below.

We summarize the systems and their duals in rational and trigonometric cases in the following table:

$$\begin{array}{ccccc} & \text{rat.CM} & \leftrightarrow & \text{rat.CM} & \\ R \rightarrow 0 & \uparrow & & \uparrow & \beta \rightarrow 0 \\ & \text{trig.CM} & \leftrightarrow & \text{rat.RS} & \\ \beta \rightarrow 0 & \uparrow & & \uparrow & R \rightarrow 0 \\ & \text{trig.RS} & \leftrightarrow & \text{trig.RS} & \end{array} \quad (15)$$

Here *CM* denotes *Calogero – Moser* models and *RS* stands for *Ruijsenaars – Schneider*. The parameters R and β here are the radius of the circle the coordinates of the particles take values in and the inverse speed of light respectively. The horizontal arrows in this

table are the quantities, relating the systems on the both sides. We notice that the duality transformations form a group which in the case of self-dual systems listed here contains $SL_2(\mathbb{Z})$. The generator S is the horizontal arrow described below, while the T generator is in fact a certain finite time evolution of the original system (which is always a symplectomorphism, which maps the integrable system to the dual one). Throughout this section q_{ij} denotes $q_i - q_j$.

Let us take trigonometric Ruijsenaars model as an example since it can be effectively described via Poissonian and Hamiltonian reductions. Let us start with the Poissonian one. Consider the space $\mathcal{A}_{\mathbf{T}^2}$ of $SU(N)$ gauge fields A on a two-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$. Let the circumferences of the circles be R and β . The space $\mathcal{A}_{\mathbf{T}^2}$ is acted on by a gauge group \mathcal{G} , which preserves a symplectic form

$$\Omega = \frac{k}{4\pi^2} \int \text{Tr} \delta A \wedge \delta A, \quad (16)$$

with k being an arbitrary real number for now. The gauge group acts via evaluation at some point $p \in \mathbf{T}^2$ on any coadjoint orbit \mathcal{O} of G , in particular, on $\mathcal{O} = \mathbb{CP}^{N-1}$. Let $(e_1 : \dots : e_N)$ be the homogeneous coordinates on \mathcal{O} . Then the moment map for the action of \mathcal{G} on $\mathcal{A}_{\mathbf{T}^2} \times \mathcal{O}$ is

$$kF_A + J\delta^2(p), \quad J_{ij} = \sqrt{-1}\nu(\delta_{ij} - e_i e_j^*) \quad (17)$$

F_A being the curvature two-form. Here we think of e_i as being the coordinates on \mathbb{C}^N constrained so that $\sum_i |e_i|^2 = N$ and considered up to the multiplication by a common phase factor.

Let us provide a certain amount of commuting Hamiltonians. Obviously, the eigenvalues of the monodromy of A along any fixed loop on \mathbf{T}^2 commute with themselves. We consider the reduction at the zero level of the moment map. We have at least $N - 1$ functionally independent commuting functions on the reduced phase space \mathcal{M}_ν .

Let us estimate the dimension of \mathcal{M}_ν . If $\nu = 0$ then the moment equation forces the connection to be flat and therefore its gauge orbits are parameterized by the conjugacy classes of the monodromies around two non-contractible cycles on \mathbf{T}^2 : A and B . Since the fundamental group $\pi_1(\mathbf{T}^2)$ of \mathbf{T}^2 is abelian A and B are to commute. Hence they are simultaneously diagonalizable, which makes \mathcal{M}_0 a $2(N - 1)$ dimensional manifold. Notice that the generic point on the quotient space has a nontrivial stabilizer, isomorphic to the maximal torus T of $SU(N)$. Now, in the presence of \mathcal{O} the moment equation implies that the connection A is flat outside of p and has a nontrivial monodromy around p . Thus:

$$ABA^{-1}B^{-1} = \exp(R\beta J) \quad (18)$$

(the factor $R\beta$ comes from the normalization of the delta-function). If we diagonalize A , then B is uniquely reconstructed up to the right multiplication by the elements of T . The potential degrees of freedom in J are "eaten" up by the former stabilizer T of a flat connection: if we conjugate both A and B by an element $t \in T$ then J gets conjugated. Now, it is important that \mathcal{O} has dimension $2(N - 1)$. The reduction of \mathcal{O} with respect to T consists of a point and does not contribute to the dimension of \mathcal{M}_ν . Thereby we

expect to get an integrable system. Without doing any computations we already know that we get a pair of dual systems. Indeed, we may choose as the set of coordinates the eigen-values of A or the eigen-values of B .

The two-dimensional picture has the advantage that the geometry of the problem suggest the $SL_2(Z)$ -like duality. Consider the operations S and T realized as:

$$S : (A, B) \mapsto (ABA^{-1}, A^{-1}); \quad T : (A, B) \mapsto (A, BA) \quad (19)$$

which correspond to the freedom of choice of generators in the fundamental group of a two-torus. Notice that both S and T preserve the commutator $ABA^{-1}B^{-1}$ and commute with the action of the gauge group. The group Γ generated by S and T in the limit $\beta, R \rightarrow 0$ contracts to $SL_2(Z)$ in a sense that we get the transformations by expanding

$$A = 1 + \beta P + \dots, \quad B = 1 + RQ + \dots$$

for $R, \beta \rightarrow 0$.

To perform the Hamiltonian reduction replace the space of two dimensional gauge fields by the cotangent space to the (central extension of) loop group:

$$T^*\hat{G} = \{(g(x), k\partial_x + P(x))\}$$

The relation to the two dimensional construction is the following. Choose a non-contractible circle \mathbf{S}^1 on the two-torus which does not pass through the marked point p . Let x, y be the coordinates on the torus and $y = 0$ is the equation of the \mathbf{S}^1 . The periodicity of x is β and that of y is R . Then

$$P(x) = A_x(x, 0), g(x) = P \exp \int_0^R A_y(x, y) dy.$$

The moment map equation looks as follows:

$$kg^{-1}\partial_x g + g^{-1}Pg - P = J\delta(x), \quad (20)$$

with $k = \frac{1}{R\beta}$. The solution of this equation in the gauge $P = \text{diag}(q_1, \dots, q_N)$ leads to the Lax operator $A = g(0)$ with R, β exchanged. On the other hand, if we diagonalize $g(x)$:

$$g(x) = \text{diag} \left(z_1 = e^{\sqrt{-1}Rq_1}, \dots, z_N = e^{\sqrt{-1}Rq_N} \right) \quad (21)$$

then a similar calculation leads to the Lax operator

$$B = P \exp \oint \frac{1}{k} P(x) dx = \text{diag}(e^{\sqrt{-1}\theta_i}) \exp \sqrt{-1}R\beta \nu r$$

with

$$r_{ij} = \frac{1}{1 - e^{\sqrt{-1}Rq_{ji}}}, i \neq j; \quad r_{ii} = - \sum_{j \neq i} r_{ij}$$

thereby establishing the duality $A \leftrightarrow B$ explicitly.

Let us briefly rephrase the discussion above in a more physical language. When Tangle Mills theory is formulated on a cylinder with the insertion of an appropriate time-like Wilson line, it is equivalent to the Sutherland model describing a collection of N particles on a circle. The observables $\text{Tr}\phi^k$ are precisely the integrals of motion of this system. One can look at other supercharges as well. In particular, when the theory is formulated on a cylinder there is another class of observables annihilated by a supercharge. One can arrange the combination of supercharges which will annihilate the Wilson loop operator. By repeating the procedure similar to the one in one arrives at the quantum mechanical theory whose Hamiltonians are generated by the spatial Wilson loops. This model is nothing but the rational Ruijsenaars-Schneider many-body system.

The self-duality of trigonometric Ruijsenaars system has even more transparent physical meaning. Namely, the field theory whose quantum mechanical avatar is the Ruijsenaars system is three dimensional Chern-Simons theory on $\mathbf{T}^2 \times \mathbf{R}^1$ with the insertion of an appropriate temporal Wilson line and spatial Wilson loop. It is the freedom to place the latter which leads to several equivalent theories. The group of (self-)dualities of this model is very big and is generated by the transformations S and T .

Let us remark that the choice of the dual Hamiltonians remains a delicate issue. At a moment we can mention two approaches to this problem. One of them exploits the embedding of the integrable many-body systems in the Toda lattice equations as the dynamics of the zeros of the corresponding τ functions [16]. In the second approach one defines the Toda lattice system in terms of the deformations of the Lagrangian manifolds of the generic Hamiltonian systems [15]. In such approach the dual Hamiltonians have a simple structure in terms of the creation operators defined in the initial system [15]. Certainly this problem deserves further investigation.

2.3 Quantum systems.

Our discussion so far concerned the classical systems. However duality is expected to be a powerful tool to deal with the quantum problems. This point has been recognized by Ruijsenaars in his early papers. The very idea is that the Hamiltonian and its dual have the common wave function and this fact could be exploited to consider one or another eigenvalue equations with respect to different variables. Since the wave functions in the systems under consideration admits the simple group like realization one could expect that the quantum duality can be formulated purely in the group theory terms too. Recent activity in this direction confirms such expectations [14].

Here we work out a few examples of quantum dual systems to illustrate the general picture. The Hamiltonian of the oscillator quantizes to:

$$\hat{H} = -1/2 \frac{\partial^2}{\partial q^2} + \frac{\omega^2 q^2}{2} \quad (22)$$

Its normalized eigen-functions are :

$$\hat{H}\psi_n = \omega(n + 1/2)\psi_n \quad (23)$$

$$\psi_n(q) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-\frac{q^2}{2\omega}}}{2^{n/2}\sqrt{n!}} H_n(q\sqrt{\omega}) \quad (24)$$

where $H_n(\xi)$ is the Hermite polynomial: $H_n(\xi) = e^{\xi^2}(-\partial_\xi)^n e^{-\xi^2}$. Using this representation of the wave-function one can easily obtain a recurrence relation :

$$\sqrt{n+1}\psi_{n+1}(q) + \sqrt{n}\psi_{n-1}(q) = \sqrt{2\omega}q\psi_n(q) \quad (25)$$

It means that $\psi_n(q)$ is an eigen-function of the following difference operator:

$$\hat{H}_D = T_+\sqrt{n} + \sqrt{n}T_-, \quad T_\pm = e^{\pm\frac{\partial}{\partial n}} \quad (26)$$

acting on the subscript n . It is easy to recognize the quantized version of the dual system.

Another example involves the Hamiltonian:

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{\nu(\nu-1)}{2\sin^2(q)} \quad (27)$$

Its normalized eigen-functions are :

$$\hat{H}\psi_n = \frac{n^2}{2}\psi_n \quad (28)$$

$$\psi_n(q) = \sin^\nu(q) \sqrt{n \frac{(n-\nu)!}{(n+\nu-1)!}} \Pi_{n-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cos(q)) \quad (29)$$

$$\Pi_l^m(x) = \frac{1}{l!} \partial_x^{l+m} \left(\frac{x^2-1}{2} \right)^l \quad (30)$$

For simplicity we take ν and n to be half-integers. One can change $\nu \rightarrow -\nu-1$ to get another eigen-function with the same eigenvalue. Using the fact that the generating function for Π_l^0 's is

$$Z(y, x) = \sum_{l=0}^{\infty} y^l \Pi_l^0 = \frac{1}{\sqrt{1-2xy+y^2}} \quad (31)$$

one derives the recurrence relations using two obvious equations:

$$(x-y)\partial_x Z = y\partial_y Z \quad (32)$$

$$(1-2xy+y^2)\partial_y Z = (x-y)Z \quad (33)$$

Next it implies:

$$(y\partial_y - m)\partial_x^m Z = (x-y)\partial_x^{m+1} Z \quad (34)$$

and hence yields:

$$((1-2xy+y^2)\partial_y + y-x)\partial_x^m Z = m(1+2y\partial_y)\partial_x^{m-1} Z \quad (35)$$

Combination of those two gives rise the desired relation.

$$x\Pi_l^m = \frac{l+1-m}{2l+1}\Pi_{l+1}^m + \frac{l+m}{2l+1}\Pi_{l-1}^m \quad (36)$$

$$\cos(q)\psi_n = \frac{1}{2}\left(\sqrt{\frac{1-\nu(\nu-1)}{n(n+1)}}\psi_{n+1} + \sqrt{\frac{1-\nu(\nu-1)}{n(n-1)}}\psi_{n-1}\right) \quad (37)$$

that is ψ_n is an eigen-function of the finite-difference operator acting on the n subscript:

$$\hat{H}_D\psi(q) = \cos(q)\psi(q) \quad (38)$$

$$\hat{H}_D = T_+\sqrt{1 - \frac{\nu(\nu-1)}{n(n-1)}} + \sqrt{1 - \frac{\nu(\nu-1)}{n(n-1)}}T_- \quad (39)$$

which is a quantum version of rational Ruijsenaars model.

Summarizing, when the system with trigonometric dependence on momentum is quantized its Hamiltonian becomes a finite difference operator. The wave functions become the functions of the discrete variables. The origin of this is in the Bohr-Sommerfeld quantization condition. Indeed, since the trigonometric dependence of momenta implies that the leaves of the polarization are compact and moreover non-simply connected the covariantly constant sections of the prequantization connection along the polarization fiber generically ceases to exist. It is only for special “quantized” values of the action variables that the section exists. In the elliptic case the quantum dual Hamiltonian is going to be a difference operator of the infinite order.

3 S-duality

Now let us explain that S-duality well established in the field theory also has clear counterpart in the holomorphic dynamical system.

The action variables in dynamical system are the integrals of meromorphic differential λ over the A -cycles on the spectral curve. The reason for the B -cycles to be discarded is simply the fact that the B -periods of λ are not independent of the A -periods. On the other hand, one can choose as the independent periods the integrals of λ over any lagrangian subspace in $H_1(\mathbf{T}_b; \mathbb{Z})$.

This leads to the following structure of the action variables in the holomorphic setting. Locally over a disc in B one chooses a basis in H_1 of the fiber together with the set of A -cycles. This choice may differ over another disc. Over the intersection of these discs one has a $Sp(2m, \mathbb{Z})$ transformation relating the bases. Altogether they form an $Sp(2m, \mathbb{Z})$ bundle. It is an easy exercise on the properties of the period matrix that the two form:

$$dI^i \wedge dI_i^D \quad (40)$$

vanishes. Therefore one can always locally find a function \mathcal{F} - *prepotential*, such that:

$$I_i^D = \frac{\partial \mathcal{F}}{\partial I^i} \quad (41)$$

The angle variables are uniquely reconstructed once the action variables are known.

To illustrate the meaning of the action-action duality we look at the two-body system, relevant for the $SU(2)$ $\mathcal{N} = 2$ supersymmetric gauge theory :

$$H = \frac{p^2}{2} + \Lambda^2 \cos(q) \quad (42)$$

with Λ^2 being a complex number - the coupling constant of a two-body problem and at the same time a dynamically generated scale of the gauge theory. The action variable is given by one of the periods of the differential pdq . Let us introduce more notations: $x = \cos(q)$, $y = \frac{p \sin(q)}{\sqrt{-2\Lambda}}$, $u = \frac{H}{\Lambda^2}$. Then the spectral curve, associated to the system which is also a level set of the Hamiltonian can be written as follows:

$$y^2 = (x - u)(x^2 - 1) \quad (43)$$

which is exactly Seiberg-Witten curve . The periods are:

$$I = \int_{-1}^1 \sqrt{\frac{x-u}{x^2-1}} dx, I^D = \int_1^u \sqrt{\frac{x-u}{x^2-1}} dx \quad (44)$$

They obey Picard-Fuchs equation:

$$\left(\frac{d^2}{du^2} + \frac{1}{4(u^2-1)} \right) \begin{pmatrix} I \\ I^D \end{pmatrix} = 0$$

which can be used to write down an asymptotic expansion of the action variable near $u = \infty$ or $u = \pm 1$ as well as that of prepotential . The duality is manifested in the fact that near $u = \infty$ (which corresponds to the high energy scattering in the two-body problem) the appropriate action variable is I (it experiences a monodromy $I \rightarrow -I$ as u goes around ∞), while near $u = 1$ (which corresponds to the dynamics of the two-body system near the top of the potential) the appropriate variable is I^D (which is actually well defined near $u = 1$ point) . The monodromy invariant combination of the periods [17]:

$$II^D - 2\mathcal{F} = u \quad (45)$$

can be chosen as a global coordinate on the space of integrals of motion . At $u \rightarrow \infty$ the prepotential has an expansion of the form:

$$\mathcal{F} \sim \frac{1}{2}u \log u + \dots \sim I^2 \log I + \sum_n \frac{f_n}{n} I^{2-4n}$$

Let us emphasize that S-duality maps the dynamical system to itself. We have seen that the notion of prepotential can be introduced for any holomorphic many-body system however its physical meaning as well as its properties deserve further investigation.

Let us consider the analogue of T duality in the Hitchin like systems [3]. It appears that the proper analogue of T duality can be identified with the separation of variables in the dynamical systems. A way of solving a problem with many degrees of freedom is to reduce it to the problem with the smaller number of degrees of freedom. The solvable models allow to reduce the original system with N degrees of freedom to N systems with 1 degree of freedom which reduce to quadratures. This approach is called a separation of variables (SoV). Recently, E. Sklyanin formulated “magic recipe” for the SoV in the large class of quantum integrable models with a Lax representation [4]. The method reduces in the classical case to the technique of separation of variables using poles of the Baker-Akhiezer function (see also [18, 19]) for recent developments and more references). The basic strategy of this method is to look at the Lax eigen-vector (which is the Baker-Akhiezer function) $\Psi(z, \lambda)$:

$$L(z)\Psi(z, \lambda) = \lambda(z)\Psi(z, \lambda) \quad (46)$$

with some choice of normalization. The poles z_i of $\Psi(z, \lambda)$ together with the eigenvalues $\lambda_i = \lambda(z_i)$ are the separated variables. In all the examples studied so far the most naive way of normalization leads to the canonically conjugate coordinates λ_i, z_i .

Remind that the phase space for the Hitchin system can be identified with the cotangent bundle to the moduli space of holomorphic vector bundle T^*M on the surface Σ . The following symplectomorphisms can be identified with the separation of variables procedure. The phase space above allows two more formulations; as the pair (C, \mathcal{L}) where C is the spectral curve of the dynamical system and \mathcal{L} is the linear bundle or as the Hilbert scheme of points on $T^*\Sigma$ where the number of points follows from the rank of the gauge group. It is the last formulation provides the separated variables. The role of Hilbert schemes on $T^*\Sigma$ in context of Hitchin system was established for the surfaces without marked points in [22] and generalized for the systems of Calogero types in [23, 24, 3].

4.1 Gaudin model

Let us present the explicit realization of the separation of variables procedure in terms of BA function in the Gaudin model. Consider the space

$$\mathcal{M} = (\mathcal{O}_1 \times \dots \times \mathcal{O}_k) // G \quad (47)$$

where \mathcal{O}_i are the complex coadjoint orbits of $G = \mathrm{SL}_N(\mathbb{C})$ and the symplectic quotient is taken with respect to the diagonal action of G .

This moduli space parameterizes Higgs pairs on \mathbb{P}^1 with singularities at the marked points $z_i \in \mathbb{P}^1$, $i = 1, \dots, k$. This is a natural analogue of the Hitchin space for genus zero. The connection to the bundles on \mathbb{P}^1 comes about as follows: consider the moduli space of Higgs pairs: $(\bar{\partial}_A, \phi)$ where ϕ is a *meromorphic* section of $\mathrm{ad}(V) \otimes \mathcal{O}(-2)$, with the restriction that $\mathrm{res}_{z=z_i} \phi \in \mathcal{O}_i$. The moduli space is isomorphic to \mathcal{M} . This space is integrable system and the separation of variables in it has been studied in [4, 21, 20]. Indeed, consider the solution to the equation

$$\bar{\partial}_A \phi = \sum_i \mu_i^c \delta^{(2)}(z - z_i) \quad (48)$$

in the gauge where $\bar{A} = 0$ which exists due to the Grothendieck's theorem on stable holomorphic bundles on \mathbf{CP}^1 . We get:

$$\phi(z) = \sum_i \frac{\mu_i^c}{z - z_i} \quad (49)$$

provided that $\sum_i \mu_i^c = 0$ and is defined up to a global conjugation by an element of G hence the Hamiltonian reduction in \mathcal{M} . Now, consider the following polynomial:

$$\text{Det}(\lambda - \phi(z)) = \sum_{i,l} A_{i,l} \lambda^i z^{-l} \quad (50)$$

The number of functionally independent coefficients $A_{i,l}$ is precisely equal to

$$k \left(\frac{N(N-1)}{2} \right) + 1 - N^2$$

In the case $N = 2$ the coadjoint orbits \mathcal{O}_i can be explicitly described as the surfaces in \mathbb{C}^3 given by the equations:

$$\mathcal{O}_i : Z_i^2 + X_i^+ X_i^- = \zeta_i^2 \quad (51)$$

with the symplectic forms:

$$\omega_i = \frac{dZ_i \wedge dX_i^+}{X_i^-} \quad (52)$$

and the complex moment maps:

$$\mu_i^c = \begin{pmatrix} Z_i & X_i^+ \\ X_i^- & -Z_i \end{pmatrix}$$

The phase space of our interest is $\mathcal{P} = \times_{i=1}^k \mathcal{O}_i // SL_2$. It is convenient to work with a somewhat larger space $\mathcal{P}_0 = \times_{i=1}^k \mathcal{O}_i / \mathbb{C}^*$, where $\mathbb{C}^* \in SL_2(\mathbb{C})$ acts as follows:

$$t : (Z_i, X_i^+, X_i^-) \mapsto (Z_i, tX_i^+, t^{-1}X_i^-)$$

The moment map of the torus \mathbb{C}^* action is simply $\sum_i Z_i$. The complex dimension of \mathcal{P}_0 is equal to $2(k-1)$. The Hamiltonians are obtained by expanding the quadratic invariant:

$$T(z) = \frac{1}{2} \text{Tr} \phi(z)^2, \quad \phi(z) = \sum_i \frac{\mu_i^c}{z - z_i} \quad (53)$$

$$\begin{aligned}
T(z) &= \sum_i \frac{\zeta_i^2}{(z - z_i)^2} + \sum_i \frac{H_i}{z - z_i} \\
H_i &= \frac{1}{2} \sum_{j \neq i} \frac{X_i^+ X_j^- + X_i^- X_j^+ + 2Z_i Z_j}{z_i - z_j}
\end{aligned} \tag{54}$$

The separation of variables proceeds in this case as follows: write $\phi(z)$ as

$$\phi(z) = \begin{pmatrix} h(z) & f(z) \\ e(z) & -h(z) \end{pmatrix}.$$

Then Baker-Akhiezer function is given explicitly by:

$$\Psi(z) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi_+ = f, \quad \psi_- = \sqrt{h^2 + ef} - h$$

and its zeroes are the roots of the equation

$$f(p_l) = 0 \Leftrightarrow \sum_i \frac{X_i^+}{p_l - z_i} = 0, \quad l = 1, \dots, k-1 \tag{55}$$

The eigenvalue $\lambda(p)$ of the Lax operator ϕ at the point p is most easily computed using the fact that $T(z) = \lambda(z)^2$. Hence, $\lambda_l = \sum_i \frac{Z_i}{p_l - z_i}$,

$$\begin{aligned}
X_i^+ &= u \frac{P(z_i)}{Q'(z_i)}, \\
Z_i &= \sum_l \frac{\lambda_l}{z_i - p_l} \frac{Q(p_l)P(z_i)}{Q'(z_i)P'(p_l)}
\end{aligned} \tag{56}$$

$$P(z) = \prod_{l=1}^{k-1} (z - p_l) \quad Q(z) = \prod_{i=1}^k (z - z_i) \tag{57}$$

The value of u can be set to 1 by the \mathbb{C}^* transformation. The (λ_l, p_l) 's are the gauge invariant coordinates on \mathcal{P}_0 . They are defined up to a permutation. It is easy to check that the restriction of the symplectic form $\sum \omega_i$ onto the set $\sum_i Z_i = 0$ is the pullback of the form

$$\sum_{l=1}^{k-1} d\lambda_l \wedge dp_l. \tag{58}$$

4.2 Points on $\mathbb{C} \times \mathbb{C}$

In this section we study the Hilbert scheme of points on $S = \mathbb{C}^2$, $S = \mathbb{C}^2/\Gamma$ for $\Gamma \approx \mathcal{I}_{ZN}, \mathcal{I}_Z$. We show that $S^{[v]}$ has a complex deformation $S_\zeta^{[v]}$ and that each $S_\zeta^{[v]}$ is an integrable model including the complexification of Calogero model.

Let us start with \mathbb{C} . As is well-known [24] it is the set of stable triples (B_1, B_2, I) , $I \in V \approx \mathbb{C}^v$, $B_1, B_2 \in \text{End}(V)$, $[B_1, B_2] = 0$ modulo the action of $\text{GL}(V)$: $(B_1, B_2, I) \sim (gB_1g^{-1}, gB_2g^{-1}, gI)$ for $g \in \text{GL}(V)$. Stability means that by acting on the vector I by arbitrary polynomials in B_1, B_2 one can generate the whole of V .

The meaning of the vector I and the operators B_1, B_2 is the following. Let z_1, z_2 be the coordinates on \mathbb{C}^2 . Let Z be a zero-dimensional subscheme of \mathbb{C}^2 of length v . It means that the space $\mathbf{H}^0(\mathcal{O}_Z)$ of functions on Z which are the restrictions of holomorphic functions on \mathbb{C}^2 has dimension v . Let V be this space of functions. Then it has the canonical vector I which is the constant function $f = 1$ restricted to Z and the natural action of two commuting operators: multiplication by z_1 and by z_2 , which are represented by the operators B_1 and B_2 . Conversely, given a stable triple (B_1, B_2, I) the scheme Z , or, rather the corresponding ideal $\mathcal{I}_Z \subset \mathbb{C}[z_1, z_2]$ is reconstructed as follows: $f \in \mathcal{I}_Z$ iff $f(B_1, B_2)I = 0$.

Now let us discuss another aspect of the space $(\mathbb{C}^2)^{[v]}$. It is symplectic manifold. To see this let us start with the space of quadruples, (B_1, B_2, I, J) with B_1, B_2, I as above and $J \in V^*$. It is a symplectic manifold with the symplectic form

$$\Omega = \text{Tr} [\delta B_1 \wedge \delta B_2 + \delta I \wedge \delta J] \quad (59)$$

which is invariant under the naive action of $G = \text{GL}(V)$. The moment map for this action is

$$\mu = [B_1, B_2] + IJ \in \text{Lie}G. \quad (60)$$

Let us perform the Hamiltonian reduction, that is take the zero level set of μ , choose a subset of stable points in the sense of GIT and take the quotient of this subset with respect to G . One can show [24] that the stability implies that $J = 0$ and therefore the moment equation reduces to the familiar $[B_1, B_2] = 0$.

Moreover, $(\mathbb{C}^2)^{[v]}$ is an integrable system. Indeed, the functions $\text{Tr} B_1^l$ Poisson-commute and are functionally independent (in open dense subset) for $l = 1, \dots, v$.

It turns out that $(\mathbb{C}^2)^{[v]}$ has an interesting complex deformation which preserves its symplecticity and integrability. Namely, instead of $\mu^{-1}(0)$ in the reduction one should take $\mu^{-1}(\zeta \cdot \text{Id})$ for some $\zeta \in \mathbb{C}$. Now $J \neq 0$. The resulting quotient $S_\zeta^{[v]}$ no longer parametrizes subschemes in \mathbb{C}^2 but rather sheaves on a non-commutative \mathbb{C}^2 , that is the “space” where functions are polynomials in z_1, z_2 with the commutation relation $[z_1, z_2] = \zeta$ [25]. Nevertheless, the quotient itself is a perfectly well-defined symplectic manifold with an integrable system on it: the functions $H_l = \text{Tr} B_1^l$ still Poisson-commute and are functionally independent for $l = 1, \dots, v$. On the dense open subset of $S_\zeta^{[v]}$ where B_2 can be diagonalized: $B_2 = \text{diag}(q_1, \dots, q_v)$ the Hamiltonians H_1, H_2 can be written as follows:

$$H_1 = \sum_i p_i, \quad H_2 = \sum_i p_i^2 + \sum_{i < j} \frac{\zeta^2}{(q_i - q_j)^2} \quad (61)$$

where $p_i = (B_1)_{ii}$. These Hamiltonians describe a collection of indistinguishable particles on a (complex) line with a pair-wise potential interaction $\frac{1}{x^2}$. This system is called rational

Calogero model . It is shown in [25] that the space B_ζ can be used for compactifying the Calogero flows in the complex case and moreover that the same compactification is natural in the KdV/KP realization of Calogero flows .

4.3 Generalization to $\mathbb{C} \times \mathbb{C}^*$ and to $\mathbb{C}^* \times \mathbb{C}^*$

Now let z_1, z_2 be the coordinates on $\mathbb{C} \times \mathbb{C}^*$, i.e. $z_2 \neq 0$. Then the description of the previous section is still valid except that B_2 must be invertible now. So in this case the Hilbert scheme of points is obtained by a complex Hamiltonian reduction from the space $T^*(G \times V)$. The moment map in our notations will be:

$$\mu = B_2^{-1} B_1 B_2 - B_1 + IJ \quad (62)$$

which corresponds to the symplectic form:

$$\Omega = \delta \text{Tr} [B_1 B_2^{-1} \delta B_2 + I \delta J] \quad (63)$$

The reduction at the non-zero level $\mu = \zeta \cdot \text{Id}$ leads to the the complex analogue of either Sutherland or rational Ruijsenaars model . In the former case $H_2 = \text{Tr} B_1^2$ while in the latter $H_{rel} = \text{Tr} (B_2 + B_2^{-1})$. On the open dense subset where B_2 diagonalizable: $B_2 = \text{diag}(\exp(2\pi i q_1), \dots, \exp(2\pi i q_v))$ the Hamiltonian H_2 equals:

$$H_2 = \sum_i p_i^2 + \sum_{i < j} \frac{\zeta^2}{\sin^2(\pi(q_i - q_j))} \quad (64)$$

Finally, if both B_1 and B_2 are invertible then we get the Hilbert scheme of points on $\mathbb{C}^* \times \mathbb{C}^*$. Its complex deformation is a bit more tricky, though. It turns out that it can be obtained via *Poisson reduction* of $G \times G \times V \times V$. The integrable system one gets in this case is the trigonometric case of Ruijsenaars model [8].

4.4 ALE models

Slightly generalizing the results of [24] one may easily present the finite-dimensional symplectic quotient construction of the Hilbert scheme of points on $T^*\mathbb{P}^1$. Take $\mathcal{V} = \mathbb{C}^v$, $A = T^*\text{Hom}(\mathcal{V}, \mathcal{V})$, $\tilde{A} = A \oplus A$ and $X = \tilde{A} \oplus T^*(\text{Hom}(\mathcal{V}, \mathbb{C}))$. The space X is acted on by the group $\mathcal{G} = GL(V) \times GL(V)$. The maximal compact subgroup \mathcal{U} of \mathcal{G} preserves the hyperkahler structure of X . The hyperkahler quotient of X with respect to \mathcal{U} is the Hilbert scheme of points on $T^*\mathbb{P}^1$ of length v .

This space is an integrable system. We shall prove it in more general setting. Namely, let S be the deformation of the orbifold $\mathbb{C}^2/\mathcal{I}_{Zk}$, where the generator $\omega = e^{\frac{2\pi i}{k}}$ of \mathcal{I}_{Zk} acts as follows: $(z_0, z_1) \mapsto (\omega z_0, \omega^{-1} z_1)$. The space $S^{[v]}$ can be described as a hyperkahler quotient. Let us take $k+1$ copy of the space \mathbb{C}^v , and denote the i 'th vector space as V_i ,

$i = 0, \dots, k$. Let us consider the space

$$X = \bigoplus_{i=0}^k \text{Hom}(V_i, V_{i+1}) \oplus \text{Hom}(V_{i+1}, V_i) \bigoplus \text{Hom}(W, V_0) \oplus \text{Hom}(V_0, W),$$

where $k+1 \equiv 0$. The space X has a natural hyperkahler structure, in particular it has a holomorphic symplectic form:

$$\omega = \text{Tr} \delta I \wedge \delta J + \sum_{i=0}^k \text{Tr} \delta B_{i,i+1} \wedge \delta B_{i+1,i} \quad (65)$$

where $B_{i,j} \in \text{Hom}(V_j, V_i)$, $I \in \text{Hom}(W, V_0)$, $J \in \text{Hom}(V_0, W)$. The space X has a natural symmetry group $G = \prod_{i=0}^k U(V_i)$ which acts on X as:

$$B_{i,j} \mapsto g_i B_{i,j} g_j^{-1}, \quad I \mapsto g_0 I, \quad J \mapsto J g_0^{-1}$$

The action of the group G preserves the hyperkahler structure of X . The complex moment map has the form:

$$\mu_i = B_{i,i+1} B_{i+1,i} - B_{i,i-1} B_{i-1,i} + \delta_{i,0} I J \quad (66)$$

The space $S^{[v]}$ is defined as a (projective) quotient of $\mu^{-1}(0)$ by the action of the complexified group G , which we denote as G_c . There is a deformation $S_\zeta^{[v]}$ which depends on k complex parameters ζ_0, \dots, ζ_k , $\sum_i \zeta_i = 0$ defined as

$$S_\zeta^{[v]} = \cap_i \mu_i^{-1}(\zeta_i \text{Id}) / G_c \quad (67)$$

Now we present the complete set of Poisson-commuting functions on $S^{[v]}$: define the “monodromy”:

$$\mathbb{B}_0 = B_{0,1} B_{1,2} \dots B_{k,0} \quad (68)$$

which transforms under the action of G in the adjoint representation. The invariants

$$f_l = \text{Tr} \mathbb{B}_0^l, \quad l = 1, \dots, v \quad (69)$$

clearly Poisson-commute on X , are gauge invariant and therefore descend to the commuting functions on $S^{[v]}$ (and to $S_\zeta^{[v]}$ as well). The functional independence is easily checked on the dense open set where $S^{[v]}$ can be identified with the symmetric product of S 's.

4.5 Integrable model of type A_n

We will consider some particular examples of the ADE or “quiver” systems which are corresponded to the case $G = Z_{n+1}$. Below we use the explicit computations for which we are thankful to A. Kotov.

Let $\epsilon = \exp(\frac{2\pi i}{n+1})$ be the generators of the group Z_{n+1} acting on \mathbb{C}^2 as $\epsilon : (q_1, q_2) \rightarrow (\epsilon q_1, \epsilon^{-1} q_2)$.

First we consider the simplest case $n = 1$. The phase space is a quotient $X = \mu^{-1}(\zeta)/G$, where μ is a momentum map of the symplectic action of $G = GL(V_0) \times GL(V_1)$ on $M = T^*Hom(V_0, V_1) \oplus T^*Hom(V_1, V_0) \oplus T^*V_0$. One can choose a local canonical system of coordinates in two ways. The first deals with the eigenvalues of the monodromy matrix $B_0 = B_{01}B_{10}$. On the dense open subset of X the eigenvalues q_i are different and nonvanishing such that $q_i = e^{\phi_i}$, $\phi_i \neq \phi_j$ for $i \neq j$. Therefore it is possible to diagonalize B_0 by $GL(V_0)$ action. Then using $GL(V_1)$ transformations we obtain $B_{01} = Q$, where $Q = diag(q_i)$ and $B_{10} = I$. Notice that the symmetry group which does not change the above mentioned condition is the diagonal subgroup $G_0 = \{(g, g) \in GL(V_0) \times GL(V_1)\}$.

Now the momentum equation

$$\begin{aligned} A_{01}B_{10} - B_{01}A_{10} + IJ &= \zeta_0 \\ A_{10}B_{01} - B_{10}A_{01} &= \zeta_1 \end{aligned}$$

has the form

$$\begin{aligned} A_{01} - QA_{10} + IJ &= \zeta_0 \\ A_{10}Q - A_{01} &= \zeta_1. \end{aligned}$$

The solution of the equation (up to a conjugation by the elements of G_0) is

$$I_i = \zeta, \quad J_i = 1, \quad (A_{10})_{ii} = \frac{p_i}{q_i}, \quad (A_{10})_{ij} = \frac{\zeta q_i}{q_i - q_j}, \quad i \neq j,$$

where $\zeta = \zeta_0 + \zeta_1$. The variables (p_i, ϕ_i) are canonical so $\omega = \sum_i dp_i \wedge d\phi_i$.

Taking the matrix $S = B_{01}A_{10}$ we see that the traces $H_k = tr S^k$ are in involution. This integrable system coincides with the Sutherland model. Namely the function $H_2 = \sum_i p_i^2 - \sum_{i < j} \frac{8\zeta^2}{sh^2(\frac{\phi_i - \phi_j}{2})}$ is the Sutherland hamiltonian.

In other hand, we can suppose that the eigenvalues of $S = B_{01}A_{10}$ are different and non-zero, then B might be diagonalized by the $GL(V_0)$ -action. Taking $B_{01} = I$ and $A_{01} = P$, $P = diag(p_i)$ we see that the momentum equations become

$$\begin{aligned} A_{01}B_{10} - P + IJ &= \zeta_0 \\ P - B_{10}A_{01} &= \zeta_1 \end{aligned}$$

and can be equivalently expressed as

$$\begin{aligned} B_{10}^{-1}PB_{10} - P + IJ &= \zeta \\ A_{01} &= B_{10}^{-1}(P - \zeta_1). \end{aligned}$$

The symplectic form $\omega = dtr A_{01} \wedge dB_{10}$ is written as

$$\omega = dtr(P - \zeta_1)dB_{10}B_{10}^{-1}.$$

It can be shown (using the arguments of [26]) that the answer leads to the rational Ruijsenaars-Schneider model. Indeed, we have

$$(A_{01})_{ij} = -\frac{\zeta e^{-\sqrt{-1}\phi_i}}{p_j + \zeta - p_i} \sqrt{\frac{P(p_i - \zeta)P(p_j + \zeta)}{-\zeta^2 P'(p_i)P'(p_j)}},$$

where $I(x) = (x-p_1)\dots(x-p_N)$ and (p_i, ϕ_i) are canonical variables. It is clear that $H_k = tr A_0^k$ are Poisson commuting integrals and $tr(A_0 + A_0^{-1})$ is a Ruijsenaars Hamiltonian.

Now it is almost straightforward to study the A_n - generalization of the model: we can introduce two different canonical systems of coordinates on the momentum solution manifold

$$\begin{aligned} A_{01}A_{10} - A_{0n}A_{n0} + IJ &= \zeta_0, \\ A_{12}A_{21} - A_{10}A_{01} &= \zeta_1, \\ &\dots \\ A_{n0}A_{0n} - A_{nn-1}A_{n-1n} &= \zeta_n \end{aligned}$$

modulo the transformations from G .

Let us diagonalize as above the monodromy matrix $A_0 = A_{01} \dots A_{n0}$ such that on the open dense subset $A_{01} = Q, Q = diag(q_i), q_i = e^{\phi_i}$ and $\phi_i \neq \phi_j$ for $i \neq j$. We can eliminate other factors by $GL(V_1), \dots, GL(V_n)$ action that means $A_{12} = A_{23} = \dots = A_{no} = I$. We are using again the diagonal symmetry group $G_0 = \{(g, g, \dots, g) \in GL(V_0) \times \dots \times GL(V_n)\}$. Therefore we obtain

$$\begin{aligned} QA_{10} - A_{0n} + IJ &= \zeta_0, \\ A_{21} - A_{10}Q &= \zeta_1, \\ A_{32} - A_{21} &= \zeta_2, \dots \\ &\dots \\ A_{0n} - A_{nn-1} &= \zeta_n \end{aligned}$$

and the result reads $(A_{10})_{ii} = \frac{p_i}{q_i}, (A_{10})_{ij} = \frac{\zeta q_i}{q_i - q_j}$. Thus the set of Hamiltonians $H_k = tr(A_{01}A_{10})^k$ yield the Sutherland systems. In particular, we have $H_2 = \sum_i p_i^2 - \sum_{i < j} \frac{8\zeta^2}{sh^2(\frac{\phi_i - \phi_j}{2})}$.

The next option is to diagonalize $A_{01}A_{10}$ such that $A_{01} = P, A_{10} = A_{21} = \dots = A_{nn-1} = I$. The momentum equations are replaced to

$$\begin{aligned} P - A_{0n}A_{n0} + IJ &= \zeta_0, \\ A_{12} - P &= \zeta_1, \\ A_{32} - A_{12} &= \zeta_2, \dots \\ &\dots \\ A_{n0}A_{0n} - A_{n-1n} &= \zeta_n \end{aligned}$$

yielding the following system:

$$\begin{aligned} P - A_{0n}PA_{0n}^{-1} + IJ &= \zeta_0, \\ A_{0n} &= (P + \zeta - \zeta_0)A_{0n}^{-1}, \\ A_{12} &= A_{n0}A_{0n} - (\zeta_2 + \dots + \zeta_n), \\ &\dots \\ A_{n-1n} &= A_{n0}A_{0n} - \zeta_n. \end{aligned}$$

The symplectic form $\omega = tr dA_{0n} \wedge dA_{n0}$ can be written as $\omega = -dtr(P + \zeta - \zeta_0)A_{0n}^{-1}dA_{0n}$. Once again we obtain that the reduced system is the Ruijsenaars model in the same line as for the case of the "deformation" of $(\mathbb{C}^2/\mathbb{Z}_2)^{[N]}$.

In this section we shall present new results concerning the identification of the proper phase space for the double elliptic system. We shall argue that two forms of Hamiltonians of the double elliptic system [12, 2] are equivalent. We follow the idea that two-body problems of Calogero-Ruijsenaars type can be attributed to the finite dimensional groups. Along this approach instead of generalization to one loop and two-loop algebras we use more complicated finite-dimensional groups. For example trigonometric Ruijsenaars model can be associated with the quantum group while the elliptic model with the Sklyanin algebra [30]. In the latter case Hamiltonian exactly coincides with the generator of the Sklyanin algebra [31].

To derive double elliptic system we need more general Poisson bracket algebra intrinsically attributed to four dimensional manifold. We have found that the proper object is the Mukai-Odesskii algebra on the intersection of the quadrics [27, 28]. The Mukai-Odesskii algebra provides a (possibly singular) Poisson structure on the intersection of n quadrics Q_i in $\mathbb{C}P^{n+2}$. The case $n = 2$ corresponds to the Sklyanin algebra. The Poisson bracket for the coordinates looks as follows [28]

$$\{x_i, x_j\} = (-1)^{i+j} \det\left(\frac{\partial Q_k}{\partial x_l}\right), l \neq i, j, \quad (70)$$

where x_i are homogeneous coordinates in $\mathbb{C}P^{n+2}$. The corresponding four dimensional manifold is "noncommutative" (in Poisson sense).

We choose the following system of four quadrics in \mathbb{C}^6 which provides the phase space for two-body double elliptic system

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ g^2 x_1^2 + x_4^2 + x_5^2 &= 1 \\ g^2 \tilde{k}^2 x_1^2 + \tilde{k}^2 x_4^2 + x_6^2 &= 1 \end{aligned} \quad (71)$$

The first pair of the equations yields the "affinization" of projective embedding of the elliptic curve into $\mathbb{C}P^3$ and the second pair provides the elliptic curve which locally is "fibered" over the first elliptic curve. If the coupling constant g vanishes the system is just the two copies of an elliptic curve embedded in $\mathbb{C}P^3 \times \mathbb{C}P^3$. Let us emphasize that the coupling constant amounts to the additional noncommutativity between the coordinates compared to the standard noncommutativity of coordinates and momenta.

The above system of the four quadrics can be considered as an affine part of a singular intersection surface $S \in \mathbb{C}P^6$:

$$\begin{aligned} x_1^2 - x_2^2 &= x_0^2 \\ x_1^2 - x_3^2 &= k^2 x_0^2 \\ g^2 x_1^2 + x_4^2 + x_5^2 &= x_0^2 \\ g^2 \tilde{k}^2 x_1^2 + \tilde{k}^2 x_4^2 + x_6^2 &= x_0^2. \end{aligned} \quad (72)$$

This is a complex singular surface i.e. two-dimensional variety.

Taking the point $(0; 0; 0; 1; 0; 0)$ outside the surface S one can consider a "stereographic projection" on \mathbb{CP}^5 such that the surface is a double-ramified covering of the following "singular $K3$ -surface" in \mathbb{CP}^5 given by the system of three quadrics:

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ \tilde{k}^2(1 - x_5^2) + x_6^2 &= 1 \end{aligned} \tag{73}$$

The relevant Poisson brackets for this particular system of quadrics read (after the proper rescaling)

$$\begin{aligned} \{x_1, x_2\} &= \{x_1, x_3\} = \{x_2, x_3\} = 0 \\ \{x_5, x_1\} &= -x_2 x_3 x_4 x_6 \\ \{x_5, x_2\} &= -x_1 x_3 x_4 x_6 \\ \{x_5, x_3\} &= -x_1 x_2 x_4 x_6 \\ \{x_5, x_4\} &= g^2 x_1 x_2 x_3 x_6 \\ \{x_5, x_6\} &= 0 \end{aligned} \tag{74}$$

The nontrivial commutation relations between coordinates on the distinct tori correspond to the standard phase space Poisson brackets while the nontrivial bracket $\{x_5, x_4\}$ means the "additional noncommutativity" of the momentum space. Let us note that the triple $x_1 x_2 x_3$ can be considered in the elliptic parametrization

$$\begin{aligned} x_1 &= \frac{1}{sn(q|k)} \\ x_2 &= \frac{cn(q|k)}{sn(q|k)} \\ x_3 &= \frac{dn(q|k)}{sn(q|k)} \end{aligned} \tag{75}$$

Let us choose x_5 as the Hamiltonian of the double elliptic system. The simple analysis shows that it is equivalent to the Hamiltonian suggested in [12]. The dual hamiltonian can be identified with $\frac{x_2}{x_1}$. When we fix a level of the Hamiltonian E and the consistency condition

$$\tilde{k}^2(1 - E^2) = 1 - c^2 \tag{76}$$

one can reduce the number of the quadrics in the system because of the coincidence of the last two equations. To get the explicit form of the spectral curve let us put $y = x_2^2 x_3^2 x_4^2$ and obtain the following equation

$$y^2 = (x_1^2 - 1)(x_1^2 - k^2)(1 - E^2 - g^2 x_1^2).$$

If we take $z = x_1^2$ we obtain that the curve under consideration is hyperelliptic of the genus 2

$$y^2 = (1 - z)(k^2 - z)(1 - E^2 - g^2 z).$$

with double branching points $1, \kappa, (1-E)/g$. We can immediately use the curve for integration of the system:

$$dt = \frac{dx_1}{x_2 x_3 x_4} = \frac{dx_1}{y},$$

$$t = \int \frac{dx_1}{\sqrt{(x_1^2 - 1)(x_1^2 - k^2)(1 - E^2 - g^2 x_1^2)}}.$$

More detailed derivation of the double elliptic systems from the Mukai-Odesskii algebras shall be presented elsewhere [29].

Let us briefly consider degenerations. If one sends \tilde{k} to zero the system degenerates to the elliptic Ruijsenaars model and the corresponding four dimensional manifold reduces to $[T^2 \times \mathbb{C}^*]_g$. At the next step one can reduce the model to the elliptic Calogero model with the manifold $[T^2 \times \mathbb{C}]_g$. Oppositely if k is sent to zero the system reduces to the one which is dual to elliptic Ruijsenaars and Calogero models respectively.

The constructions involving quadrics provide very explicit description of the noncommutative manifolds above. Indeed let us consider limit \tilde{k} which amounts to the system of three quadrics in $\mathbb{C}P^5$

$$\begin{aligned} x_1^2 - x_2^2 &= 1 \\ x_1^2 - x_3^2 &= k^2 \\ g^2 x_1^2 + x_4^2 + x_5^2 &= 1. \end{aligned} \tag{77}$$

This system corresponds to the K3 manifold which is the phase space of the elliptic Ruijsenaars system. Geometrically it looks as the cylinder bundled over the elliptic curve. The general discussion on the completely integrable systems associated with K3 manifolds can be found in [13].

Let us discuss the relation with the Sklyanin algebra [30] visible in this respect geometrically. To this aim let us interpret the Poisson bracket relations in the Sklyanin algebra

$$\{S_\alpha, S_0\} = 2J_{\alpha,\beta} S_\beta S_\gamma \tag{78}$$

$$\{S_\alpha, S_\beta\} = 2S_0 S_\gamma \tag{79}$$

as example of the Mukai-Odesskii algebra generated by two quadrics in $\mathbb{C}P^4$. The quadrics

$$K_1 = \sum_{n=1}^3 S_n^2, K_2 = S_0^2 + \sum_{n=1}^3 J_n S_n^2 \tag{80}$$

coincide with the center of the Poisson bracket algebra. Hence the Sklyanin algebra fits with the general scheme.

Now let us remind the observation made in [31] that the Hamiltonian of the elliptic Ruijsenaars system coincides with the generator of the Sklyanin algebra S_0 . On the other hand moving along our approach the elliptic Ruijsenaars Hamiltonian coincides with coordinate x_5 . This means that in the limit of vanishing \tilde{k} the system of four

quadrics in \mathbb{CP}^4 effectively reduces to a system of two quadrics in \mathbb{CP}^4 and two different Hamiltonians for elliptic Ruijsenaars model linear in coordinates can be identified. The degeneration to the trigonometric Ruijsenaars model can be performed in the similar way and the corresponding Hamiltonian can be expressed in terms of the coordinates which follow from the realization of $U_q(SL(2))$ in terms of the intersection of two quadrics in \mathbb{CP}_4 . Some generalization of this picture for the trigonometric many-body system can be found in [32].

Let us emphasize that it is clear from analysis above that the discussed form of the double elliptic system is not the general one. Indeed one could simply consider more general form of the system of quadrics to generate the multiparametric Poisson bracket. As an example one could introduce the "momentum coupling constant" \tilde{g} and generate the system which is totally dual with respect to the change of coordinate and momenta as well as two coupling constants.

For completeness let us note that the periodic Toda two-body system also can be described in terms of the quadratic algebra. To this aim one can consider the following Poisson algebra

$$\begin{aligned} \{A_1, A_2\} &= 2A_3(4 - A_2) \\ \{A_3, A_2\} &= A_2 \\ \{A_1, A_3\} &= A_1 + A_3^2 \end{aligned} \tag{81}$$

Then the Hamiltonian of the Toda system is linear function

$$H_{Toda} = 1/2(A_1 + A_2) \tag{82}$$

This picture can be considered as the further degeneration from the elliptic Calogero model via Inozemtzev limit.

4.7 Brane interpretation

In the brane terms separation of variables can be formulated as reduction to a system of D0 branes on some four dimensional manifold. It reminds a reduction to a system of point-like instantons on a (generically noncommutative [25]) four manifold. One more essential point is that separated variables amount to some explanation of the relation of periodic Toda chain above and monopole chains. Indeed, monopole moduli space have the structure resembling the one for the Toda chain in separated variables; both of them are the Hilbert schemes of points on the similar four manifolds.

The abovementioned constructions of the separation of variables in integrable systems on moduli spaces of holomorphic bundles with some additional structures can be described as a symplectomorphism between the moduli spaces of the bundles (more precisely, torsion free sheaves) having different Chern classes.

To be specific let us concentrate on the moduli space $\mathcal{M}_{\vec{r}}$ of stable torsion free coherent sheaves \mathcal{E} on S . Let $\hat{A}_S = 1 - [\text{pt}] \in H^*(S, \mathbb{Z})$ be the A -roof genus of S . The vector

$\vec{v} = \text{Ch}(\mathcal{E}) \vee \text{As} = (r, w, u - r) \in H^*(S, \mathbb{Z})$, $w \in H^2(S, \mathbb{Z})$, corresponds to the sheaves with the Chern numbers:

$$ch_0(\mathcal{E}) = r \in H^0(S; \mathbb{Z}) \quad (83)$$

$$ch_1(\mathcal{E}) = \vec{w} \in H^2(S; \mathbb{Z}) \quad (84)$$

$$ch_2(\mathcal{E}) = d \in H^4(S; \mathbb{Z}) \quad (85)$$

Type *IIA* string theory compactified on S has BPS states, corresponding to the Dp -branes, with p even, wrapping various supersymmetric cycles in S , labelled by $\vec{v} \in H^*(S, \mathbb{Z})$. The actual states correspond to the cohomology classes of the moduli spaces $\mathcal{M}_{\vec{v}}$ of the configurations of branes. The latter can be identified with the moduli spaces $\mathcal{M}_{\vec{v}}$ of appropriate sheaves.

The string theory, compactified on S has moduli space of vacua, which can be identified with

$$\mathcal{M}_A = O(\Gamma^{4,20}) \backslash O(4, 20; \mathbb{R}) / O(4; \mathbb{R}) \times O(20; \mathbb{R})$$

where the arithmetic group $O(\Gamma^{4,20})$ is the group of discrete automorphisms. It maps the states corresponding to different \vec{v} to each other. The only invariant of its action is \vec{v}^2 .

We have studied three realizations of an integrable system. The first one uses the non-abelian gauge fields on the curve Σ imbedded into symplectic surface S . Namely, the phase space of the system is the moduli space of stable pairs: (\mathcal{E}, ϕ) , where \mathcal{E} is rank r vector bundle over Σ of degree l , while ϕ is the holomorphic section of $\omega_{\Sigma}^1 \otimes \text{End}(\mathcal{E})$. The second realization is the moduli space of pairs (C, \mathcal{L}) , where C is the curve (divisor) in S which realizes the homology class $r[\Sigma]$ and \mathcal{L} is the line bundle on C . The third realization is the Hilbert scheme of points on S of length h , where $h = \frac{1}{2} \dim \mathcal{M}$.

The equivalence of the first and the second realizations corresponds to the physical statement that the bound states of N $D2$ -branes wrapped around Σ are represented by a single $D2$ -brane which wraps a holomorphic curve C which is an N -sheeted covering of the base curve Σ . The equivalence of the second and the third descriptions is natural to attribute to T -duality.

Let us mention that the separation of variables above provides some insights on the Langlands duality which involves spectrum of the Hitchin Hamiltonians. The attempt to reformulate Langlands duality as a quantum separation of variables has been successful for the Gaudin system corresponding to the spherical case [21]. The consideration in [2] suggests that the proper classical version of the Langlands correspondence is the transition to the Hilbert scheme of points on four-dimensional manifold. This viewpoint implies that quantum case can be considered as correspondence between the eigenfunctions of the Hitchin Hamiltonians and solutions to the Baxter equation in the separated variables.

5 Discussion

We would like to conclude with some general remarks concerning to the application of the dualities above to other moduli spaces. The systems of the Hitchin type are closely related

to the instantons and monopole moduli on the manifolds with the compact dimensions. Namely the Hitchin system on the torus without marked points can be mapped by T-duality transformation along two dimensions onto the the moduli space of instantons on $\mathbb{R}^2 \times T^2$ [33]. When the marked point is added then T-duality transforms it to the noncommutativity of the manifold where instantons are considered on and the Hilbert scheme on the noncommutative manifold emerges. That is what we have described at another language via the separations of variables procedure.

In a similar manner the relation to the monopole moduli can be formulated. The Hilbert schemes enters the description of the monopole moduli spaces very naturally. It is well known that moduli space of monopoles on \mathbb{R}^4 in commutative theory is geometrically $Hilb\mathbb{C} \times \mathbb{C}^*$ which on the Hitchin side corresponds to the system on a cylinder. Recently it was argued that the Hitchin theory on the cylinder with marked points is mapped via a chain of dualities into the moduli space of $SU(2)$ monopoles on $\mathbb{R}^3 \times S^1$ [34] with prescribed asymptotics.

Finally let us comment on six dimensional theory on NS5 branes compactified on a three dimensional torus \mathbf{T}^3 down to three dimensions. As was discussed extensively in [35] in case where two out of three radii of \mathbf{T}^3 are much smaller then the third one \mathbf{R} the effective three dimensional theory is a sigma model with the target space \mathcal{X} being the hyper-kahler manifold (in particular, holomorphic symplectic) which is a total space of algebraic integrable system. The complex structure in which \mathcal{X} is the algebraic integrable system is independent of the radius \mathbf{R} while the Kähler structure depends on \mathbf{R} in such a way that the Kähler class of the abelian fiber is proportional to $1/\mathbf{R}$. In the limit of the large \mathbf{R} theory is effectively four-dimensional and the dualities between integrable systems have the dualities between the Coulomb moduli spaces as the field theory counterparts. This issue has been reviewed in [36].

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